Optimal Multi-Period Spectrum Model for the Measurement of Random Behaviour of Assets Returns

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Abstract

Options have become extremely popular and the reasons behind that can be summarized in two points; they are attractive tools both for speculation and hedging. If their price can be determined: therefore their trading can be done with a certain confidence.The vendor of the option have two mains questions. How much should the buyer of the option pay in other words, how to access the price at the time t = 0 and the richness available at time T ?becomes the pricing problem. Multi fractals offer a well-defined set of answers to this question because it has the capability of generating various degree of long term memory in different powers of return. A model cannot capture all aspects of reality but rather a simple version that focuses on some particular point of interest. We present a dynamic multi-period spectrum model of variation of the capital market price aimed at determining the growth rate of an asset, using a continuous rate of return, $r_t = -e^{-\gamma\alpha}$*; and the optimal trading strategy.*

Key Words: *Dynamic Multi-period, Spectrum Model, Capital Market, Trading Strategy and Asset Return*

INTRODUCTION

Let $(\mathbb{R}^n, \beta(\mathbb{R}^n))$; be a measurable space and let $f: \beta(\mathbb{R}^n) \to \mathbb{R}$ be a measurable function. Then the multi-fractal spectrum is defined by

$$D(\alpha) = \{\varepsilon \epsilon \beta(\mathbb{R}^n): \bar{f}(\alpha)\} \leq \Delta\alpha; \varepsilon = \alpha \qquad (1.1)$$

The basic problem is to calculate the function $D(\alpha)(Sun\ etal\ 2001)$.To do this, we need to know what the function $\bar{f}(\alpha)$ is. The multi-fractal formalism of multi-affine functions amounts to compute the spectrum,$D(\alpha)$, define a set where the fractal exponent is equal to α. (Xiao 2004)

Following

equations $D_h(u, x) = \lim_{r \to 0} sup \frac{\mu(\bar{B}(x,r))}{h(r)}$ and

$\lim_{t\to 0} \sup \frac{X_t^d}{h(t)} \leq \frac{(1+\gamma)^2}{2\theta} = \frac{\left(1+\frac{1}{\bar{f}_{\alpha(0,1)\Delta\alpha}}\right)^2}{2\theta}$. to obtain the function $f(\alpha) = \lim_{r\to 0} sup \frac{T(r)}{h(r)}$ in our case, we require the local asymptotic behavior of the sample path of the process. And what comes to mind is the subject of the law of iterated logarithm (LIL).To this end, we assume a double stochastic integrals by a direct adaptation of the case of the Brownian motion and set (Uzoma 2006)

$$h(t) = 2t \log log \frac{1}{t}, \text{ for } t > 0. \quad (1.2)$$

In what follows, we now state;

Lemma 1.1

For $t > 0$and $h(t) = 2t \log log \frac{1}{t}$, the singularity spectrum $D(\alpha)$ defined as the Hausdorff dimension of the set where the fractal exponent is equal to α (Xiong 2002) is given by

$$\lim_{t\to 0} sup \frac{X_t^d}{h(t)} = \frac{(1+\gamma)^2}{2\theta}, \theta = 1, \gamma\epsilon[0,1], \quad (1.3a)$$

where$\gamma = \frac{1}{\bar{f}_{\alpha(0,1)\Delta\alpha}}$. (1.3b)

Proof

Let d be a predictable process valued in a bounded interval $[\alpha_0, \alpha_1]$ for some real parameters, $0 \leq \alpha_0 \leq \alpha_1$, and$X_t^d \coloneqq \int_0^t \int_0^u a_r dW_r dW_u$.

Then $\alpha_0 \leq lim_{t\to 0} \frac{2X_t^d}{h(t)} \leq \alpha_1$ $a.s.$

Now set $\bar{\alpha} = \frac{(\alpha_0+\alpha_1)}{2} \geq 0$

and

$\delta \coloneqq \frac{(\alpha_1+\alpha_0)}{2}$.

For the first inequality, we have by the law of the iterated logarithm for the Brownian motion.

$$\bar{\alpha} = \lim_{t\to 0} sup \frac{2X_t^d}{h(t)} \leq \delta \lim_{t\to 0} sup \frac{2X_t^d}{h(t)} + \lim_{t\to 0} sup \frac{2X_t^d}{h(t)} ,$$

Where$\check{d} = \delta^{-1}(\bar{\alpha} - d)$ is the value in$[-1,1]$. It then follows from the second inequality that;

$$\lim_{t\to 0} sup \frac{2X_t^d}{h(t)} \geq \bar{\alpha} - \delta = \beta_0.$$

For the proof of the second inequality, we assume without loss of generality that $\|d\|_\infty \leq 1$.

Let $T > 0$ and $\lambda > 0$ be such that$2\lambda T < 1$. Then from Doob's maximal inequality for sub-martingales for all $\beta \geq 0$, we have

$$P\left[\max_{0\leq t\leq T} 2X_t^d \geq \beta\right] = P\left[\max_{0\leq t\leq T} exp\left(2\lambda X_t^d\right) \geq exp(\lambda\beta)\right]$$

$$\leq e^{-\lambda\beta} E\left[e^{2\lambda X_T^d}\right]$$

$$= e^{-\lambda(\beta+T)}(1-2\lambda T)^{-1/2}. \quad (1.4)$$

Take $\theta, \gamma\epsilon(0,1)$ and set all $K\epsilon N$

$$\beta_k = (1+\gamma)^2 h(\theta^k)$$

and

$\lambda_k = [2\theta^k(1+\gamma)]^{-1}$.

Applying

$$\mu\left(\bar{B}(0,r)\right) = \left|\left\{E\epsilon\mathbb{R}^n : X_t^{d,n} \epsilon \bar{B}(0,r)\right\}\right|$$

$$= \int_0^\infty I_{\bar{B}(0,r)} X_t^{d,n} dt$$

$$= T(r).$$ we have that for all $K\epsilon N$,

$$P\left[\max_{0\leq t\leq \theta^k} 2X_t^d \geq (1+\gamma)^2 h(\theta^k) \leq e^{-1/2(1+r)}(1+r^{-1})(-k\, log\, \theta)^{-(1+\gamma)}\right].$$

It follows from the Borel-Cantelli lemma that for almost all $w\epsilon\Omega$ and since

$$\sum_{k>0} \frac{1}{K^{(1+\gamma)}},$$

there exists a natural number $K^{\theta,\gamma}(w)$ such that for all $k \geq K^{\theta,\gamma}(w)$,

$$\max_{0\leq t\leq \theta^k} 2X_t^d(w) < (1+\gamma)^2 h(\theta^k).$$

In particular, for all $t \in (\theta^{k+1}, \theta^k)$,

$2X_t^d(w) < (1+\gamma)^2 h(\theta^k) \leq (1+\gamma)^2 \frac{h(t)}{\theta}$.

Hence

$$\lim_{t\to 0}\sup\frac{X_t^d}{h(t)} \le \frac{(1+\gamma)^2}{2\theta} = \frac{\left(1+\frac{1}{\bar{f}_{\alpha(0,1)\Delta\alpha}}\right)^2}{2\theta}. \qquad (1.5)$$

1.2 The general case

In the real world in general, markets are neither ideal nor complete (Val and Zaka 2004). Therefore, the equation

$\Delta = \frac{\partial V}{\partial S} \pm H$(1.6a)

can be seen as real world behavior of a stock market price.

Consider hedging a market comprising h unit of asset in long position and one unit of the option in short position. At time T the market value is assumed to be $h - P(E) \pm H$ or $hP - W \pm H$, following (1.6a).

After an elapse, Δt, the value of the market will change by an amount/rate $h(\Delta P + D_1\Delta t) - \Delta W$,in view of the dividend received on h unit held, where D_1 is the market price of risk, following

$M_D(\bar{f}_\alpha) = \lim_{\varepsilon\to 0}\sum_k r_k^D\,; D > 0\,; r_k < \varepsilon$. By Ito's lemma we have

$$h(uP\Delta t + \sigma P\Delta z + D_1\Delta t) = \left[\left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P}uP + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2 P^2\right)\Delta t + \frac{\partial W}{\partial P}\sigma P\Delta z\right] \pm H$$

or

$$\left[huP + hD_1 - \left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P}uP + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2 P^2\right)\right]\Delta t = \left[h\sigma P - \frac{\partial W}{\partial P}\sigma P\right]\Delta z H. \qquad (1.6b)$$

Take $h = \frac{\partial W}{\partial P}$, then the uncertainty term disappears and the market in this case is temporarily riskless (no signal). It should therefore grow in value by the riskless rate in force i.e.

$$\left(huP + hD_1 - \left[\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P}uP + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2 P^2\right]\right)\Delta t \pm H$$
$$= \quad (hP - W)\gamma\Delta t \pm H \qquad (1.7)$$

Thus,

$$D_1\frac{\partial W}{\partial P} - \left(\frac{\partial W}{\partial t} + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2 P^2\right) = \left(\frac{\partial W}{\partial P}P - W\right)\gamma, \qquad (1.8)$$

so that

$$\frac{\partial W}{\partial t} + (\gamma P - D_1)\frac{\partial W}{\partial P} + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2 P^2 = \gamma W \ . \qquad (1.9)$$

Under the following dynamics

$$dP_t = \alpha(t)P_t dt + \sigma(S_t)P_t dW(t),$$

where $\alpha(t) = \ln\left(\frac{P_{t+\Delta t} - P_t}{\Delta t}\right)$ is the rate of stock price changes at time t.

we have the version of the parabolic partial differential equation (4.9) with $D_1 = 0$ as;

$$-\frac{\partial W(P,t)}{\partial t} = rP\frac{\partial W(P,t)}{\partial P} + \frac{1}{2}\sigma^2(S_t)P^2\frac{\partial^2 W(P,t)}{\partial P^2} \pm H$$

$$-rW(P,t) \pm H, \forall (P,t) \in (0,\infty) \times (0,T), \qquad (1.10a)$$

Following equation

$$y_w(\tau) - y_0(\tau) = \frac{\partial V}{\partial S} \pm (H_o + H_w) \pm$$

H_o. $\quad y_w(\tau) - y_0(\tau) = \frac{\partial V}{\partial S} \pm (2H_o)$we have;

$$-\frac{\partial W(P,t)}{\partial t} = rP\frac{\partial W(P,t)}{\partial P} + \frac{1}{2}\left(\frac{\sigma_m^2}{\left(1 + f(\delta(t,T)\lambda^2 S^2 \Gamma / \bar{f}\Delta\alpha)\right)}\right)(S_t)P^2\frac{\partial^2 W(P,t)}{\partial P^2} \pm (H_o + H_w) \pm H_o$$
(1.10b)

and

$$= rP\frac{\partial W(P,t)}{\partial P} + \frac{1}{2}\left(\frac{\sigma_m^2}{\left(1 + f(\delta(t,T)\lambda^2 S^2 \Gamma / \bar{f}\Delta\alpha)\right)}\right)(S_t)P^2\frac{\partial^2 W(P,t)}{\partial P^2} \pm 2H_o \qquad (1.10c)$$

1.3 The optimal trading strategy

Let us denote an optimal trading strategy π_t^* for which we define

$$H_{\pi_t^*}(t,\gamma,P) = E_{\pi_t^*}[U(W(T))]. \qquad (1.11)$$

Our objective here is to find the optimal value function such that

$$H(t,\gamma,,P) = \sup_{\pi_t^* \in \pi} H_{\pi_t^*}(t,\gamma,,P). \qquad (1.12)$$

Assume equation (4.9) for $D_1 \neq 0$ together with the optimal π_t^*, we have

$$\frac{\partial W}{\partial t} + (\gamma P - D_1)\frac{\partial W}{\partial P} + \frac{1}{2}\frac{\partial^2 W}{\partial P^2}\sigma^2(S_t)\pi_t^* P^2 = \gamma W. \qquad (1.13)$$

Put $z = \frac{\alpha}{P}$; $\quad W(P) = z^{\beta} H(Z)$. Since $W$is dependent on γ, differentiating and substituting into equation (1.13), gives

$$\gamma z^{\beta} H = \frac{\sigma^2(S_t)}{2}\left[\beta(\beta+1)z^{\beta}H + \beta z^{\beta+1}\frac{\partial H}{\partial z} + 2(\beta+1)z^{\beta+1}\frac{\partial H}{\partial z} + z^{\beta+2}\pi_t^*\frac{\partial^2 H}{\partial z^2}\right] + \left(\frac{\gamma\alpha}{z} - D_1\right)\left(-\frac{1}{\alpha}\right)\left(\beta z^{\beta+1}H + z^{\beta+2}\frac{\partial H}{\partial z}\right). \tag{1.14}$$

Now cancelling by z^{β} and collecting like terms give

$$0 = \frac{\sigma^2(S_t)\pi_t^*}{2} z^2 \frac{\partial^2 H}{\partial z^2} + \frac{\partial H}{\partial z} z\left(\sigma^2(\beta+1) - \gamma - \frac{D_1}{\alpha} z\right)$$

$$+H\left(\frac{\sigma^2}{2}\beta(\beta+1) - \gamma, (\beta+1) + \beta\frac{D_1}{\alpha} z\right). \tag{1.15}$$

Set $\frac{\sigma^2(S_t)}{2}\beta = r$ and let $\frac{D_1/\alpha}{\frac{\sigma^2(S_t)}{2}} = -1$ to have

$\beta = \frac{2r}{\sigma^2(S_t)}, \alpha = -\frac{2D_1}{\sigma^2(S_t)}$. SubstitSuteintoequation (4.14) to obtain

$$z\pi_t^* H_{zz} + (2 - z)H_z = \beta(H - H_z). \tag{1.16}$$

Solving for optimal trading strategy, we have

$$\pi_t^* = \frac{\beta(H - H_z)}{zH_{zz}} + \frac{(z-2)H_z}{zH_{zz}}$$

$$= \frac{\frac{2r}{\sigma^2(S_t)}(H - H_z)}{zH_{zz}} + \frac{(z-2)H_z}{zH_{zz}}$$

$$= \frac{\frac{2\frac{\sigma^2(S_t)}{2}\beta}{\sigma^2(S_t)}(H - H_z)}{zH_{zz}} + \frac{(z-2)H_z}{zH_{zz}}. \tag{1.17}$$

$$= \frac{\frac{2\left(\frac{\left(\frac{\sigma_m^2}{\left(1+f\left(\delta(t,T)\lambda^2 S^2 \Gamma/\bar{f}\Delta\alpha\right)\right)}\right)(S_t)}{2}\right)\beta}{\sigma^2(S_t)}(H - H_z)}{zH_{zz}} + \frac{(z-2)H_z}{zH_{zz}}$$

The model has some successes in explaining excess volatility of stock returns compared to fundamentals and negative Skewness of equity returns as well as generating multi-fractal spectrum. By equation (4.6) there is no market signal as it tends to zero, meaning that the market is likely to crash at such point, signifying insolvency in asset returns.

$$f(\alpha) = \lim_{k\to\infty}\left(\frac{\ln N(\alpha)}{\ln s^k}\right). \tag{1.18}$$

Denoting by $N(\alpha, \Delta t)$ the number of intervals $[t, t + \Delta t]$ required to cover $\tau(\alpha)$ we can write equation (1.27b)~ $S_t = S(t, T, r) = Ke^{-r(T-t)}$

$$N(\alpha, \Delta t) \sim (\Delta t)^{-f(\alpha)}. \tag{4.19}$$

The market prices correspond to the values of α between α_{min} and $\alpha(f_{max})$

REFERENCES

Erin Pearse.(2000), An introduction to dimension theory and fractal geometry: fractal Dimensions and Measures.(google)

Sun, H. Chen, Z. Wu and Y. Yuan,(2001) Multifractal analysis of Hang Seng index in Hong Kong stock market, Physica A. 291： 553–562.

Taylor, S.J., (1967). On the connection between Hausdorff measure and generalized capacity. *Provo Cambridge Philos. Soc.*,57 :524-531.

Uzoma, P. U., (2006). An estimate of the exact size of the singular points of Laplace equation in Euclidian space (a case study of the earthling screen).*An M.Sc. thesis in mathematics department. Abia state University, Uturu, Nigeria.*

Valeri. I. and Zakamouline,(2004), Efficient Analytic Approximation of the Optimal Hedging Strategy for European Call Option with Transaction Costs. Stochastic Finance (2004), at Autumn School and International Conference.

Xiao, Y., (2004). Random fractals and Markov processes in fractal geometry and application. (A Jubilee of Benoit Mandelboirot M.L. Laipidus and Frankenhuijsen, Eds. 5:261-338.

Xiong Zhang-feng.(2002). Estimating the Fractal Dimension of Financial Time Series by Wavelet, Systems Engineering-theory & Practice, 12: 48-53, 122.